Generalized Thermostatistics and Bose-Einstein Condensation

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Abstract

Analytical expressions for Bose-Einstein condensation of an ideal Bose gas analyzed within the strictures of non-extensive, generalized thermostatistics are here obtained.

Nonextensive thermostatistics (NEXT) [1, 2, 3, 4] has been utilized with success in connection with a number of problems both in the classical [5, 6, 7, 8, 9, 10, 11] and the quantum regimes [12, 13, 14, 15, 16]. NEXT is thought to be relevant for the study (among others) of: systems described by non linear Fokker-Planck equations [5]; systems with a scale-invariant occupancy of phase space [6]; non equilibrium scenarios involving temperature fluctuations [7, 8]; systems exhibiting weak chaos [9, 10]; and systems with interactions of long range relative to the system's size [11]. This list is far from complete. For reviews on the applications of the NEXT formalism see [1, 2, 4].

The applications of non-extensive thermostatistics to the classical domain are much more developed than applications to quantum mechanical problems. The (comparatively) slow progress made in applying NEXT ideas to quantum systems may be due, at least in part, to the considerable difficulty in obtaining analytical results concerning basic (statistical) aspects of quantum many-body physics. The aim of this letter is to report on one such result, in connection with the phenomenon of Bose-Einstein condensation (BEC). The recent successful experimental realization of BEC in atomic gases [18, 19, 20] has led to considerable theoretical and experimental activity [12, 21, 22, 23, 24]. In point of fact, some of the theoretical efforts have focused on the role of the statistics on BEC [12] particularly when changing from the extensive one based upon the Boltzmann-Gibbs (BG) logarithmic entropy, to the generalized version proposed in Ref. [3].

Within the standard Boltzmann-Gibbs' thermostatistical formalism, an ideal Bose-Einstein gas is the simplest exactly solvable continuous system that undergoes a phase transition. Generalized non-extensive statistics is characterized by a non-logarithmic entropy S_q that contains a free parameter q called the non-extensivity index (for $q \to 1$, one has $S_q \to$ Boltzmann's S). Application of Jaynes' MaxEnt methodology [25] within a NEXT-context yields power-law probability distributions [26]. Previous S_q-studies of BEC have only been undertaken in restricted regions [12] in the vicinity of the $q \to 1$ limit where BG-statistics is recovered. In this communication we considerably amplify this rather restricted scope by generating NEXT-exact analytic expressions, and then determine the all-important $q-$values for which BEC may occur.

Consider an ideal gas of particles of mass m in d dimensions which obey Bose-Einstein statistics. In the BG case the average number of particles in the grand canonical ensemble

is given by

$$N_d^{BG} = c_d \int_0^\infty \frac{p^{d-1}}{z^{-1}e^{\beta\epsilon} - 1} dp \tag{1}$$

$$= c_d \left(\frac{1}{2}\right)^d \left(\frac{2m}{\beta}\right)^{\frac{d}{2}} \int_0^\infty \frac{x^{\frac{d}{2}-1}}{z^{-1}e^x - 1} dx \tag{2}$$

$$= c_d \left(\frac{1}{2}\right)^d \left(\frac{2m}{\beta}\right)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right) g_{\frac{d}{2}}(z) \tag{3}$$

where $z = e^{\beta\mu}$, $\epsilon = \frac{p^2}{2m}$, $\beta = \frac{1}{T}$, μ is the chemical potential, $g_\nu(z) = \frac{1}{\Gamma(\frac{d}{2})} \int_0^\infty \frac{x^{\frac{d}{2}-1}}{z^{-1}e^x - 1} dx = \Sigma_{j=1}^\infty \frac{z^j}{j^\nu}$ is the well known Bose-Einstein integral [17] and c_d is a constant which depends on the intergration over the other phase space variable of dimension d. Since occupation numbers cannot be negative or infinite, $\mu < 0$ and $0 \leq z < 1$. For the limiting value $z \to 1$, $g_\nu(z) \to \zeta(\nu)$ where $\zeta(x)$ is the Riemann zeta function and hence only for d=3 does Bose-Einstein condensation occur.

In the case where NEXT is employed, the average number of particles is given by

$$N_d^{TS} = c_d \int_0^\infty \frac{p^{d-1}}{\left[1 + (q-1)\beta(\frac{p^2}{2m} - \mu)\right]^{\frac{1}{q-1}} - 1} dp \tag{4}$$

$$= c_d \int_0^\infty \frac{p^{d-1}}{(a + bp^2)^{\frac{1}{q-1}} - 1} dp \tag{5}$$

where

$$a = 1 - (q-1)\mu\beta$$

and

$$b = \frac{(q-1)\beta}{2m}.$$

Here we have used the power-law distribution function for bosons [16]. For $\mu < 0$ and $q > 1$ equation(5) may be written as

$$N_d^{TS} = c_d \int_0^\infty \Sigma_{n=0}^\infty \frac{p^{d-1}}{(a + bp^2)^{\frac{n+1}{q-1}}} dp \tag{6}$$

$$= c_d \Sigma_{n=0}^\infty \frac{(\frac{a}{b})^{\frac{d}{2}}}{a^{\frac{n+1}{q-1}}} \int_0^\infty \frac{x^{d-1}}{(1 + x^2)^{\frac{n+1}{q-1}}} dx \tag{7}$$

$$= c_d \Sigma_{n=0}^\infty \frac{\left(\frac{2m(1-(q-1)\mu\beta)}{(q-1)\beta}\right)^{\frac{d}{2}}}{(1 - (q-1)\mu\beta)^{\frac{n+1}{q-1}}} \frac{\frac{1}{2}\Gamma(\frac{d}{2})\Gamma(\frac{n+1}{q-1} - \frac{d}{2})}{\Gamma(\frac{n+1}{q-1})} \tag{8}$$

provided $0 < d < 2\frac{n+1}{q-1}$. For $\mu \to 0$

$$N_d^{TS} \to c_d \left(\frac{2m}{(q-1)\beta}\right)^{\frac{d}{2}} \frac{1}{2}\Gamma\left(\frac{d}{2}\right) \Sigma_{n=0}^{\infty} \frac{\Gamma(\frac{n+1}{q-1} - \frac{d}{2})}{\Gamma(\frac{n+1}{q-1})}. \tag{9}$$

In the limit $\mu \to 0$ the sum in eq(9) is only convergent for d=3 since for large z, $\frac{\Gamma(z-a)}{\Gamma(z)} \to \frac{1}{z^a}$. Furthermore in this case $q < 5/3$.

Hence, utilizing NEXT, rather than BG, does not permit BEC to occur in systems with dimensionality less than 3. In 3 dimensions BEC ocurs both in the BG and NEXT treatments, in the latter case only for $1 < q < 5/3$.

HGM acknowledges the hospitality of the Theoretical Physics Institute of the Department of Physics at the University of Alberta and the Physics Department of SUNY at Fredonia where part of this work was undertaken.

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